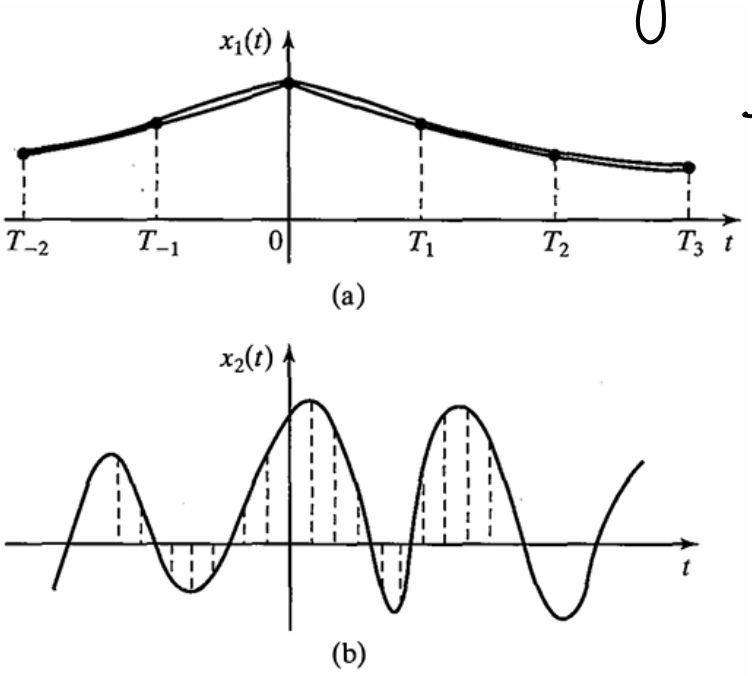
Analog to digital conversion:

7 1.1 : Sampling theorem :

Consider 2 signals $x_1(t)$ and $x_2(t)$



→ smooth signal which varies slowly, main frequency content is at low frequencies

↳ signal changes rapidly due to presence of high frequency component

Smooth signals with low-frequency changes ($x_1(t)$) don't need to be sampled very often. A larger sampling interval (T_1) is sufficient to capture their information because they don't change rapidly.

Signals with rapid, high-frequency changes ($x_2(t)$) must be sampled much more frequently to avoid losing information. A smaller sampling interval (T_2) is needed.

The statement says that if a signal $x(t)$ has a maximum frequency of W (it's "bandlimited" to W), meaning its frequency content is zero for all frequencies greater than or equal to W, then it is sufficient to sample it with a sampling interval of $T_s = \frac{1}{2W}$.

What It Means

- **Bandlimited Signal:** This means the signal doesn't contain any frequencies higher than a certain value, W. A typical human voice, for instance, has a maximum frequency of about 4 kHz, so it's bandlimited to $W = 4$ kHz.
- **Sampling Interval (T_s):** This is the time between consecutive samples. A smaller T_s means you're taking samples more frequently.
- **The Rule ($T_s = \frac{1}{2W}$):** This formula tells us the **maximum** time we can wait between samples to still be able to perfectly reconstruct the original signal. The reciprocal of this, $f_s = \frac{1}{T_s} = 2W$, is known as the **Nyquist rate**. It's the minimum sampling frequency required.

Sampling Theorem:

Let signal $x(t)$ have a bandwidth W i.e., $X(f) \equiv 0$ for $|f| \geq W$. Sampling interval T_s, $T_s \leq \frac{1}{2W}$ to yield sequence $\{x(nT_s)\}_{n=-\infty}^{n=\infty}$

$x(t)$ from sampled values,

$$x(t) = \sum_{n=-\infty}^{\infty} 2W'T_S\, x(nT_S)\, \mathrm{sinc}\left[2W'(t-nT_S)\right]$$

$$W \le W' \le \frac{1}{T_S} - W$$

In special case, $T_S = \frac{1}{2W}$, $W' = W = \frac{1}{2T_S}$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_S)\, \mathrm{sinc}\left(\frac{t}{T_S} - n\right)$$

$x_\delta(t)$ denote result of sampling original signal by impulse at nT_S.

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_S)\, \delta(t - nT_S)$$

$$x_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) .$$

property made $x(nT_s) = x(t)$

FT on b.s,

$$X_\delta(f) = X(f) * F\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

$$X_\delta(f) = X(f) * \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{T_s}\right)$$

Convolution property,

$$X(f) * \delta\left(f - \frac{n}{T_s}\right) = X\left(f - \frac{n}{T_s}\right)$$

Filter transfer func :

$$H(f) = T_s \Pi\left(\frac{f}{2W'}\right)$$

↓ rectangular

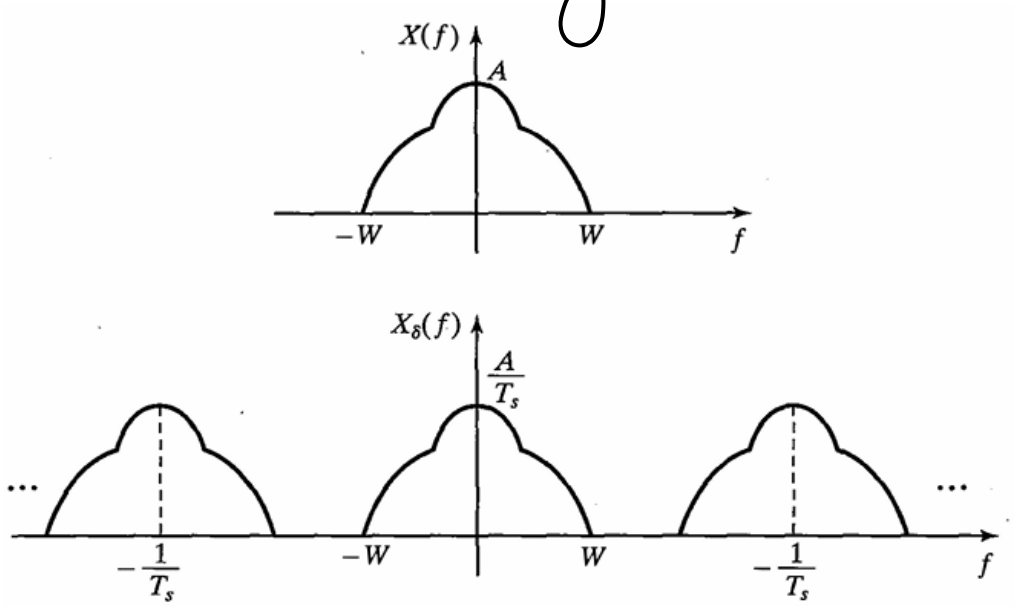


Figure 7.2 Frequency-domain representation of the sampled signal.

$$X(f) = X_\delta(f)\, T_s \Pi\left(\frac{f}{2W'}\right)$$

IFT,

$$x(t) = x_\delta(t) \circledast 2W' T_s \operatorname{sinc}(2W't)$$

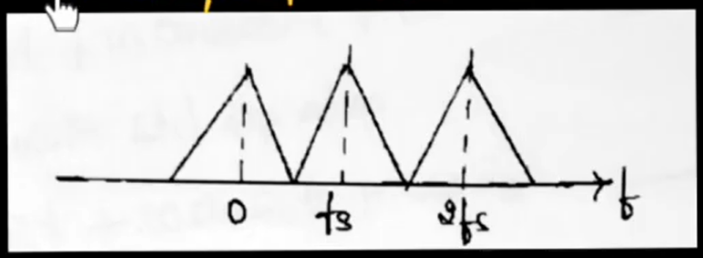
$$x(t) = \sum_{n=-\infty}^{\infty} 2W' T_s\, x(nT_s) \operatorname{sinc}(2W'(t - nT_s))$$

$$W' = W = \frac{1}{2T_s},$$

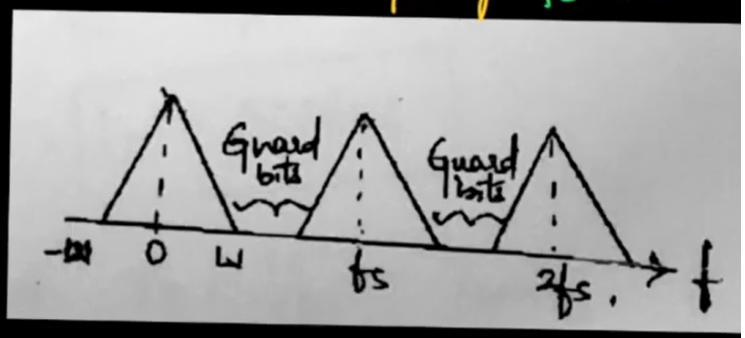
$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \mathrm{sinc}\,(t - nT_s)$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

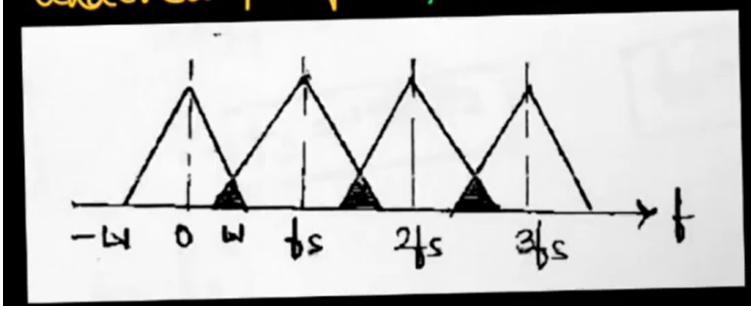
Correct Sampling $f_s = 2W$



Over Sampling $f_s > 2W$



Under Sampling $f_s < 2W$



Example 7.1.1

In this development, we have assumed that samples are taken at multiples of T_s. What happens if we sample regularly with T_s as the sampling interval, but the first sample is taken at some $0 < t_0 < T_s$?

Instead of nT_s, 1st sample is

$$t = t_0 + nT_s.$$

$t = nT_s$ (samples taken @ multiple of T_s)

Reconstructed signal,

$$y(t) = \sum_{n=-\infty}^{\infty} y(nT_s)\, \mathrm{sinc}\left[2W'(t - nT_s)\right]$$

→ Shannon interpolation formula

New signal,

$$x(t + t_0) = \sum_{n=-\infty}^{\infty} x(t_0 + nT_s)\, \mathrm{sinc}\left(2W(t - nT_s)\right)$$

Sub $t = -t_0$ gives,

$$x(0) = \sum_{n=-\infty}^{\infty} x(t_0 + nT_s)\, \mathrm{sinc}\left[2W(t_0 + nT_s)\right]$$

Example 7.1.2

A bandlimited signal has a bandwidth equal to 3400 Hz. What sampling rate should be used to guarantee a guard band of 1200 Hz?

$W_G = f_s - 2W$

guard band

$1200 = f_s - 2(3400)$

$f_s = 1200 + 6800$

$\boxed{f_s = 8000\,Hz}$

7.2 : Quantization

- After sampling we have discrete time signals with values at nT_s.
- We do quantisation inorder to bring amplitude discrete as well.

7.2.1 : Scalar quantization.

- Quantization is rounding process.
- In scalar quantization, the set of real no. being partitioned into N disjoint subsets denoted by R_k, $1 \leq k \leq N$ (each called quantisation regions), each region have a quanization value (→ denoted $\hat{x}_i$) to which all value in that particular region is mapped.

- After quantisation, we encode. $\log_2 N$ bits are enough to encode these levels in binary sequence.
- The no. of bits required to transmit each source o/p is $R = \log_2 N$ bits.
- Quantisation leads to distortion.
- Consider an example of 8-level quantisation:

$a_2 = a_1 + \Delta$

$a_3 = a_2 + 2\Delta$

..

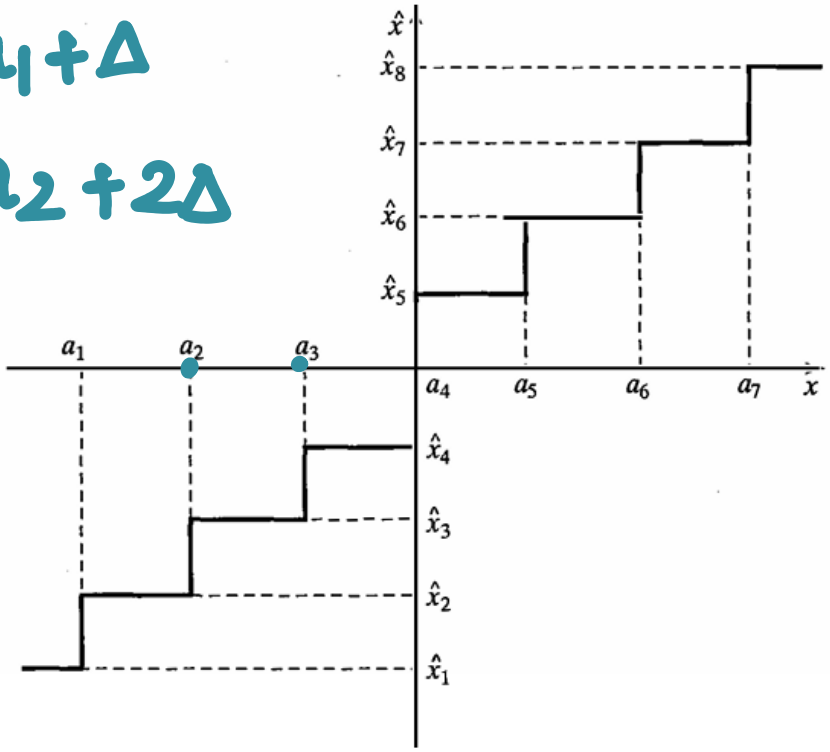


Figure 7.3 Example of an eight-level quantization scheme

8 level so 8 regions,

Binary equivalent

$R_1 = (-\infty, a_1] \longrightarrow 000$

$R_2 = (a_1, a_2] \longrightarrow 001$

$R_3 = (a_2, a_3] \longrightarrow 010$

$R_4 = (a_3, a_4] \longrightarrow 011$

$R_5 = (a_4, a_5] \longrightarrow 100$

$R_6 = (a_5, a_6] \longrightarrow 101$

$R_7 = (a_6, a_7] \longrightarrow 110$

$R_8 = (a_7, \infty) \longrightarrow 111$

$$\Delta(\text{Step size}) = \frac{\text{Amplitude}_{max} - \text{Amplitude}_{min}}{\text{no. of level required}}$$

used to define region with this Δ

The quantised function $Q(x)$ defined by,

$$Q(x) = \hat{x}_i \quad \forall \; x \in R_i$$

$\frac{max + min}{2}$

Quantised value not give exact signal thus result in distortion called quantised noise denoted $\tilde{x}$ (also called error)

$$\tilde{x}[n] = X[n] - Q(X)$$

min. error = 0 ; max. error = $\frac{\Delta}{2}$

No. of levels $(N) = 2^n$ → bits required for sample

Squared error distortion measure,

$$d(x, \hat{x}) = (x - Q(x))^2 = \tilde{x}^2$$

Mean squared error,

$$D = E(X - Q(X))^2$$

∴ Normal power can be calculated by

$$P = \frac{1}{T} \int_{-T}^{T} x(t)^2 \, dt$$

If given signal is random then

$$P = \int_{-\infty}^{\infty} \text{Power spectral density}$$

$$= \int_{-\infty}^{\infty} S_x(f)\, df = R_{xx}(\tau)$$

$$S_x(f) = F.T\left[R_x(\tau)\right]$$

$$\hookrightarrow E\left(X(t_1)\,X(t_2)\right)$$

$$\boxed{\tau = t_2 - t_1}$$

Gaussian Random process,

mean = 0 ; variance = power

UNIFORM QUANTISATION.

Example 7.2.1

The source $X(t)$ is a stationary Gaussian source with mean zero and power spectral density

$$S_x(f) = \begin{cases} 2 & |f| < 100 \text{ Hz} \\ 0 & \text{otherwise} \end{cases}.$$

The source is sampled at the Nyquist rate and each sample is quantized using the eight-level quantizer which is shown in Figure 7.3 This figure has $a_1 = -60$, $a_2 = -40$, $a_3 = -20$, $a_4 = 0$, $a_5 = 20$, $a_6 = 40$, $a_7 = 60$, and $\hat{x}_1 = -70$, $\hat{x}_2 = -50$, $\hat{x}_3 = -30$, $\hat{x}_4 = -10$, $\hat{x}_5 = 10$, $\hat{x}_6 = 30$, $\hat{x}_7 = 50$, and $\hat{x}_8 = 70$. What is the resulting distortion and rate?

rate means the average number of bits used per sample to represent the quantized values after encoding

$$\text{Bit Rate} = (\text{Bits/sample}) \times (\text{Samples/second}) = 3 \times f_s$$

$f_s = 2(100) = 200$ Hz.

Bit Rate = no. of bits $\times$ f_s

$= \log_2 N \times f_s = \log_2(8) \times f_s$

$= \dfrac{\log N}{\log 2} \times f_s = 3 \times 200$

To calculate $\log_2 N$

$= 600$ bits/sec.

variance,

$$\sigma^2 = E(X_i^2) = R_X(\tau)\Big|_{\tau=0}$$

$$= \int_{-\infty}^{\infty} S_x(f)\,df$$

$$= \int_{-100}^{100} 2\,df = 400.$$

Distortion,

$$D = E(x-\hat{x})^2 = \int_{-\infty}^{\infty} (x-Q(x))^2 f_x(x)\,dx$$

↓ PDF

$$D = \sum_{i=1}^{8} \int_{R_i} (x-Q(x))^2 f_x(x)\,dx$$

$\simeq 33.38$ (using Q-func.)

when zero o/p,

$$D = E(x-0)^2 = E(x^2) = 400\%$$

Here $f_x(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$f_x(x) = \frac{1}{\sqrt{2\pi(400)}} e^{-\frac{x^2}{800}}$$

gaussian func.

. Effectiveness of Quantization (Reduction in Distortion)

- By using **3 bits per sample** (giving 8 possible quantization levels), the distortion is improved to **33.38**.
- The **improvement factor** is:

$$\frac{400}{33.38} \approx 11.98$$

That's how many times lower the distortion is compared to always guessing zero.

- In decibels (dB), this reduction is:

$$10 \log_{10}(11.98) \approx 10.78 \text{ dB}$$

mean of Random variable,

(continuous), $E(X) = \int x f_x(x)\, dx$

R_1, $\int_{-\infty}^{a_1} (x - Q(x))^2 f_x(x)\, dx$

$\vdots$

R_8, $\int_{a_7}^{\infty} (x - Q(x))^2 f_x(x)\, dx$.

R_2 quantisation $= \dfrac{a_1 + a_2}{2}$ $\rightarrow a_1 + \Delta$

$= a_1 + \dfrac{a_1 + \Delta}{2}$

$= a_1 + \dfrac{\Delta}{2}$.

Distortion (D) $= \frac{\Delta^2}{12}$

(For uniform quantisation).

•) If Q value of R_1 given 3.25

If " " R_2 " 3.75

Find boundary of each region

$\Delta = 3.75 - 3.25 = 0.5$

max error $= \frac{\Delta}{2}$.

R_1 boundary $\left(3.25 - \frac{\Delta}{2}, 3.25 + \frac{\Delta}{2}\right)$

R_2 " $\left(3.75 - \frac{\Delta}{2}, 3.75 + \frac{\Delta}{2}\right)$

•) SQNR (signal to noise ratio) given as,

$$SQNR = \frac{E(X^2)}{E(X - Q(X))^2}.$$

Quantisation noise power is,

$$P_{\tilde{X}} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} E(X(t) - Q(X(t)))^2 \, dt$$

Signal power,

$$P_X = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} E(X^2(t)) \, dt$$

$$SQNR = \frac{P_X}{P_{\tilde{X}}}$$

Example 7.2.2

Determine the SQNR for the quantization scheme given in Example 7.2.1.

$P_X = 400$, $P_{\tilde{X}} = D = 33.38$

$$SQNR = \frac{P_X}{P_{\tilde{X}}} = 11.98 = 10 \log 11.98$$

$\simeq 10.78$ dB.

$10 \log_{10}$:

Used when calculating the ratio of powers or energies (such as distortion, noise power, or signal power).

$20 \log_{10}$:

Used when calculating the ratio of amplitudes (such as voltage, current, or pressure) because power is proportional to the square of amplitude.

Non uniform Quant : (less constrains so better quantizer than uniform quant)

$$\hat{x}_i = \frac{\int_{a_{i-1}}^{a_i} x f_X(x)\,dx}{\int_{a_{i-1}}^{a_i} f_X(x)\,dx}$$

$$D = \int_{-\infty}^{a_1} (x - \hat{x}_1)^2 f_X(x)dx + \sum_{i=1}^{N-2} \int_{a_i}^{a_{i+1}} (x - \hat{x}_{i+1})^2 f_X(x)dx + \int_{a_{N-1}}^{\infty} (x - \hat{x}_N)^2 f_X(x)dx.$$

Gaussian source with mean m and variance σ^2 is used, then the values of a_i and $\hat{x}_i$ read from Table 7.2 are replaced with $m + \sigma a_i$ and $m + \sigma \hat{x}_i$, respectively, and the value of the distortion D will be replaced by $\sigma^2 D$.

TABLE 7.2 OPTIMAL NONUNIFORM QUANTIZER FOR A GAUSSIAN SOURCE

N	$\pm a_i$	$\pm \hat{x}_i$	D	$H(\hat{X})$
1	—	0	1	0
2	0	0.7980	0.3634	1
3	0.6120	0, 1.224	0.1902	1.536
4	0, 0.9816	0.4528, 1.510	0.1175	1.911
5	0.3823, 1.244	0, 0.7646, 1.724	0.07994	2.203
6	0, 0.6589, 1.447	0.3177, 1.000, 1.894	0.05798	2.443
7	0.2803, 0.8744, 1.611	0, 0.5606, 1.188, 2.033	0.04400	2.647
8	0, 0.5006, 1.050, 1.748	0.2451, 0.7560, 1.344, 2.152	0.03454	2.825
9	0.2218, 0.6812, 1.198, 1.866	0, 0.4436, 0.9188, 1.476, 2.255	0.02785	2.983
10	0, 0.4047, 0.8339, 1.325, 1.968	0.1996, 0.6099, 1.058, 1.591, 2.345	0.02293	3.125

Example 7.2.3

How would the results of Example 7.2.1 change if, instead of the uniform quantizer shown in Figure 7.3, we used an optimal nonuniform quantizer with the same number of levels?

$a_i = \sigma \times a_i'$ values.

$\sigma^2 = 400 \Rightarrow \sigma = 20$.

Boundary values,

$a_1 = -a_8 = -20 \times 1.748 = -34.96$

$a_2 = -a_7 = -20 \times 1.050 = -21$

$a_3 = -a_6 = -20 \times 0.5006 = -10.012$

$a_4 = 0 = a_5$

Quantized values,

$\hat{x}_1 = -\hat{x}_8 = -20 \times 2.152 = -43.04$

$\hat{x}_2 = -\hat{x}_7 = -20 \times 1.344 = -26.88$

$$\hat{x}_3 = -\hat{x}_6 = -20 \times 0.756 = -15.12$$

$$\hat{x}_4 = -\hat{x}_8 = -20 \times 0.2451 = -4.902$$

$D = 0.03454$ from Table 7.2 &

$$D_{\text{Non-uni}} = \sigma^2 D = 400(0.03454) = 13.816$$

$$SQNR = \frac{400}{13.816} = 28.95$$

$$\simeq 14.62 \text{ dB}$$

Key Take away: 3.45 dB better than SQNR of uniform quant.

7.2.2: Vector Quantization

↓

To reduce the error.

•) Consider 2 sample at source at each time, these 2 samples as a point in plane, quantizer partitions entire plane into 16 quantization regions.

4 level quant.

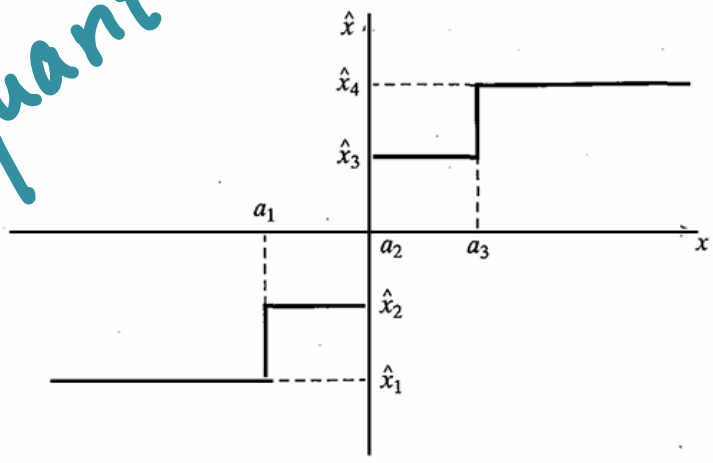
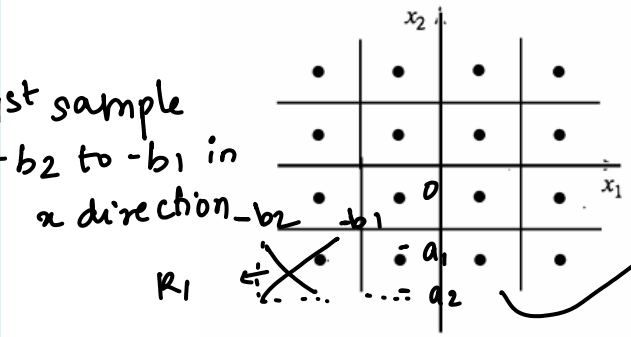


Figure 7.4 A four-level scalar quantizer.



1st sample $-b_2$ to $-b_1$ in x direction

2nd sample $-a_2$ to a_1 in y direction

Figure 7.5 A scalar four-level quantization applied to two samples

• 3-D (3 samples) result the space into 64 regions.

• n length → n dimensions

• Rate = $\frac{\log K}{n}$ bits/source o/p.

K ← quantised values.

7.4: Wave Code Techniques:

Continous Signal → To → Binary signal

7.4.1: Pulse Code Modulation.

•) PCM has 3 basic sections S Q E (sampler, quantizer, encoder).

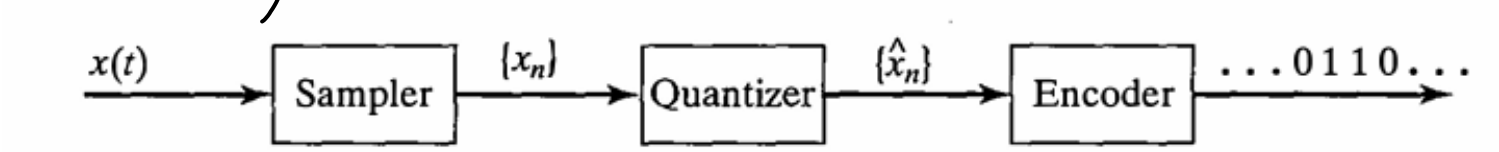


Figure 7.7 Block diagram of a PCM system.

PCM we make assumptions:

(i) The signal is bandlimited with a max. frequency of W. Therefore it can be fully reconstructed from samples taken at rate $f_s = 2W$ or higher.

(ii) Signal is of finite amplitude

x_{max} $\forall$ t, we have $|x(t)| < x_{max}$

(iii) Quantization done with large no. of levels $N = 2^n$.

•) Before entering sampler signal sent to bandlimited filter to prevent any component beyond W. The filter is called presampling filter.

UNIFORM PCM: (Quantizer assumed to be uniform).

•) The range of i/p samples $[-x_{max}, x_{max}]$ no. of quantization levels, length of each region

$$\Delta = \frac{2x_{max}}{N} = \frac{x_{max}}{2^{n-1}}$$

•) Error $(\tilde{x}) = x - Q(x)$, take random value $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$

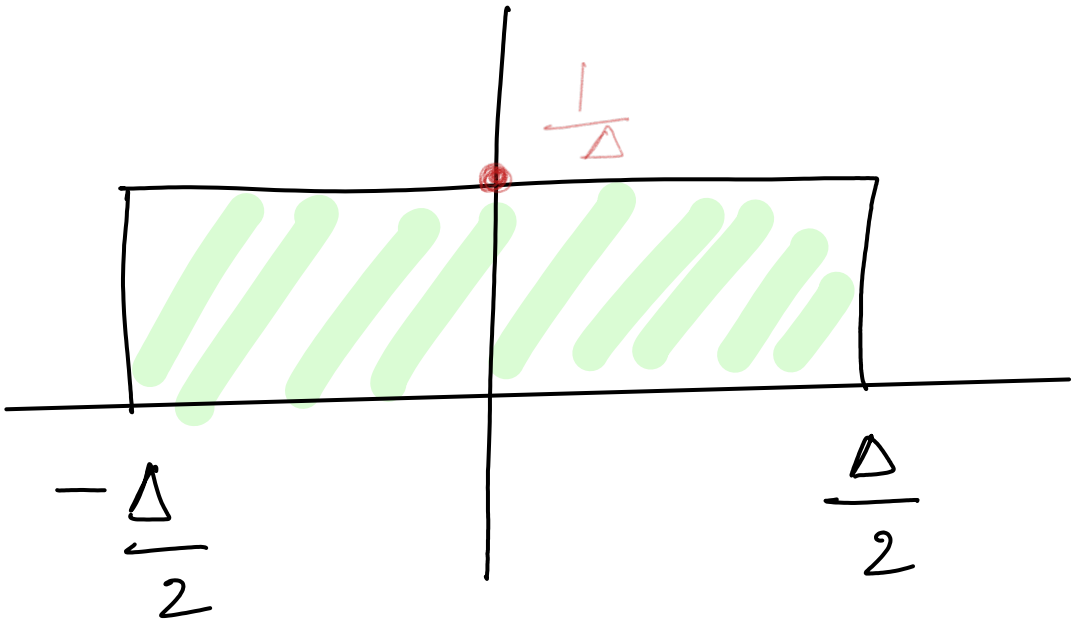
•) In PCM, N high and variation in i/p signal amplitude is small. i.e.,

$\Delta \rightarrow$ small.

such that error is uniformly distributed random variable.

$$f(\tilde{x}) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} \leq \tilde{x} \leq \frac{\Delta}{2} \\ 0, & \text{otherwise.} \end{cases}$$

:) error is equiprobable.



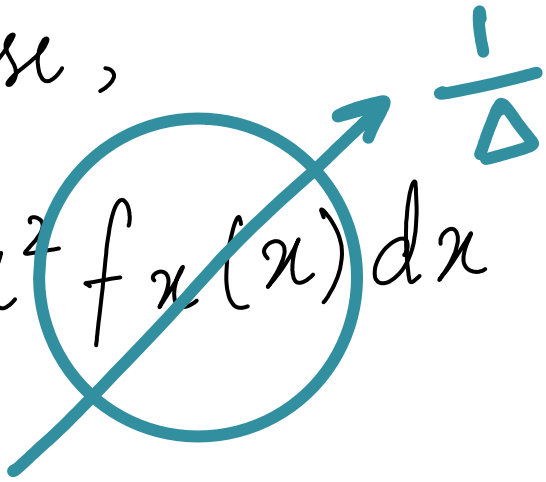
area of $\square$ gle $= l \times b$

$$= \frac{1}{\Delta}\left(\frac{\Delta}{2} + \frac{\Delta}{2}\right)$$

$$= \frac{1}{\Delta}\left(\frac{2\Delta}{2}\right) = 1.$$

Quantisation noise,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx$$



$$E[\tilde{X}^2] = \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta}\tilde{x}^2\,d\tilde{x}$$

$$= \frac{1}{\Delta}\frac{\tilde{x}^3}{3}\Big|_{-\Delta/2}^{\Delta/2}$$

$$= \frac{1}{\Delta}\left[\left(\frac{\Delta}{6}\right)^3 - \left(-\frac{\Delta}{6}\right)^3\right]$$

$$= \frac{1}{\Delta}\left[\frac{\Delta^3}{24} + \frac{\Delta^3}{24}\right]$$

$$= \frac{1}{\Delta}\left[\frac{\Delta^3}{24}\right]^2$$

$$= \frac{1}{12}$$

noise power,

$$E[X^2] = \frac{\Delta^2}{12} = \frac{4x_{max}^2}{12N^2}$$

$$= \frac{x^2_{max}}{3N^2}$$

$$= \frac{x^2_{max}}{3(2^n)^2}$$

$$E[x^2] = \frac{x^2_{max}}{3\times 4^n}.$$

$$SQNR = \frac{P_X}{\tilde{x}^2} = \frac{3 \times 4^n}{x_{max}^2} \cdot P_X .$$

when $X(t)$ is wide-sense stationary, P_X

$$P_X = R_X(\tau)\Big|_{\tau=0}$$

$$= \int_{-\infty}^{\infty} S_x(f)\, df$$

$$= \int_{-\infty}^{\infty} x^2 f_x(x)\, dx$$

- $P_X \leq x_{max}^2$, so $P_X / x_{max}^2 < 1$ (usually, the average power is much less than the maximum possible).

SQNR Equation in dB

$$\mathrm{SQNR_{dB}} \approx 10 \log_{10} \left(\frac{P_X}{x_{max}^2} \right) + 6\eta + 4.8$$

- 6ν: Means each extra bit (ν) doubles the number of levels, which increases the SQNR by about 6 dB per bit.
- $10 \log_{10}(P_X / x_{max}^2)$: Reflects the relationship between the average power and maximum signal level.
- 4.8: A constant term from the derivation for uniform quantizers.

SQNR derivation:

$$SQNR|_{in\ dB} = 10\log_{10}\left(\frac{3 \times 4^n P_X}{x^2_{max}}\right)$$

$$= 10\log_{10}(3) + 10\log_{10}(4)^n + 10\log_{10}\left(\frac{P_X}{x^2_{max}}\right)$$

$$= 4.77 + n(6.020) + 10\log_{10}\left(\frac{P_x}{x^2_{max}}\right)$$

$$n = \log_2(256) = 8$$

$$SQNR = 4^8 = 65536 \simeq 48.16\,dB$$

• If signal with Bandwidth W, samples required for perfect reconstruction is 2W samples/sec. Some guard band required (f_s) which is > 2W. For n bits used, total nfs bit/sec required for Tx.

• Bit Rate (R) = 2nW bits/sec = nfs

• BW for PCM = $\frac{R}{2}$ = nW.

NON-UNIFORM PCM: (signals like speech).

- Speech signal has high probability of smaller amplitude & smaller probability of higher amplitude. Make quantizer with more regions of smaller amplitude & less " " higher ".

- Compressor is first step where amplification of small amplitude & attenuation of high " take place.

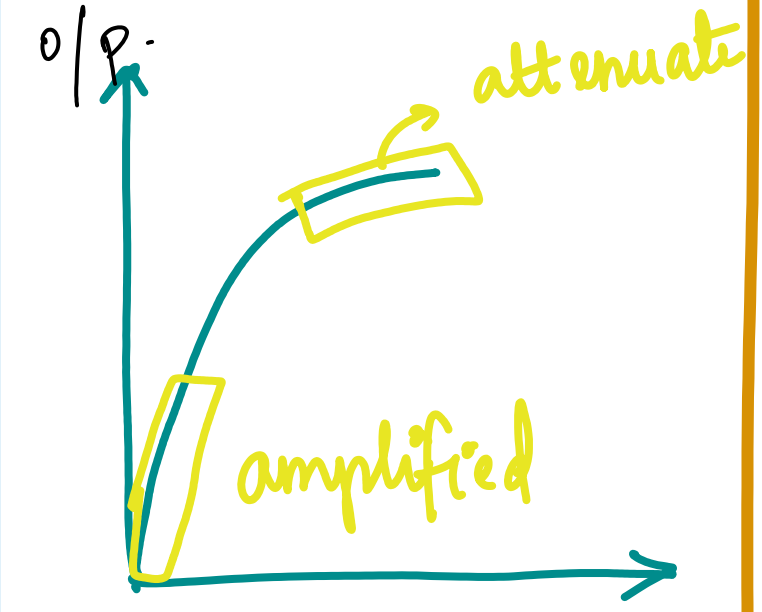
✓ Passes to uniform PCM.

✓ Then expansion where inverse operation of compressor takes place.

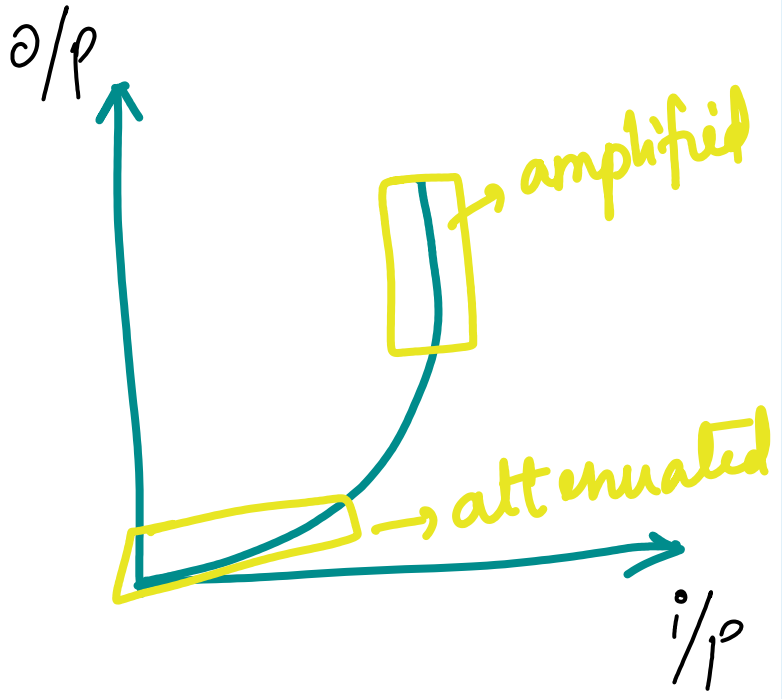
:) Above 3 pts combining called Companding technique.

Characteristics

Compressor



Expander



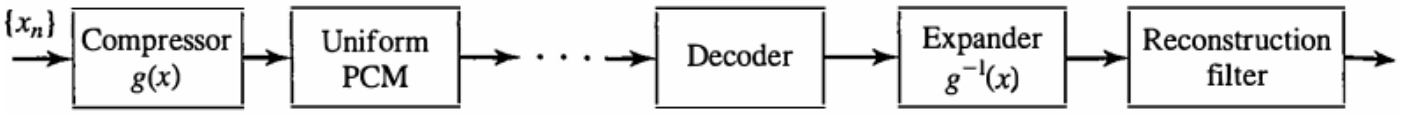


Figure 7.8 Block diagram of a nonuniform PCM system.

2 types of companders

(i) Z

μ-law compander, $|x| \leq 1$

$$g(x) = \frac{\log(1+\mu|x|)}{\log(1+\mu)} \operatorname{sgn}(x)$$

$\mu = 255$ followed by uniform PCM by 8 level quantisation

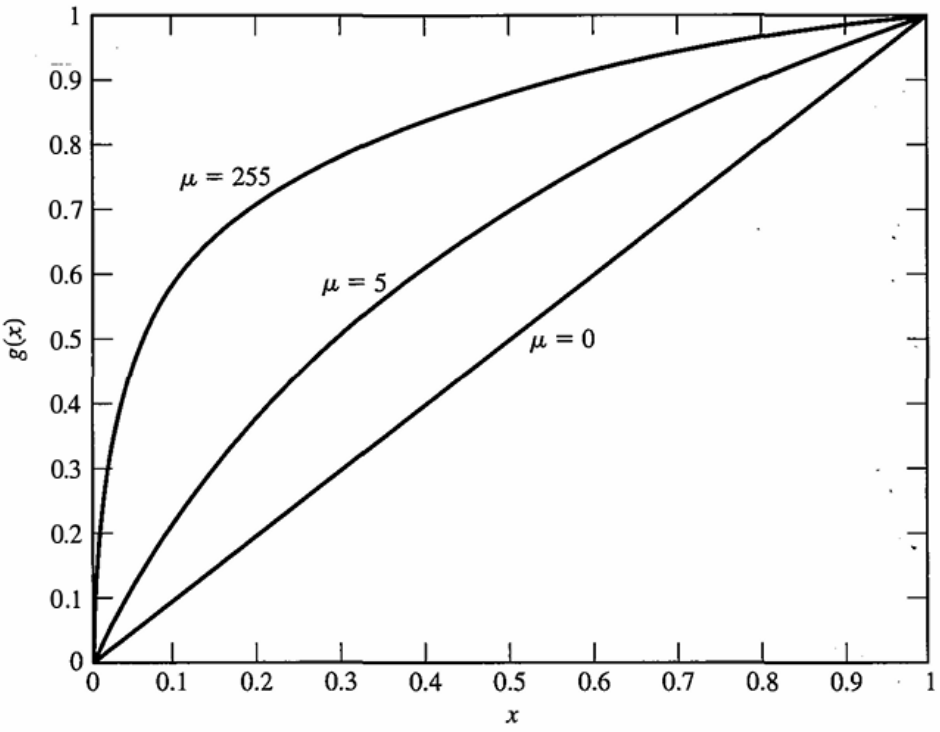
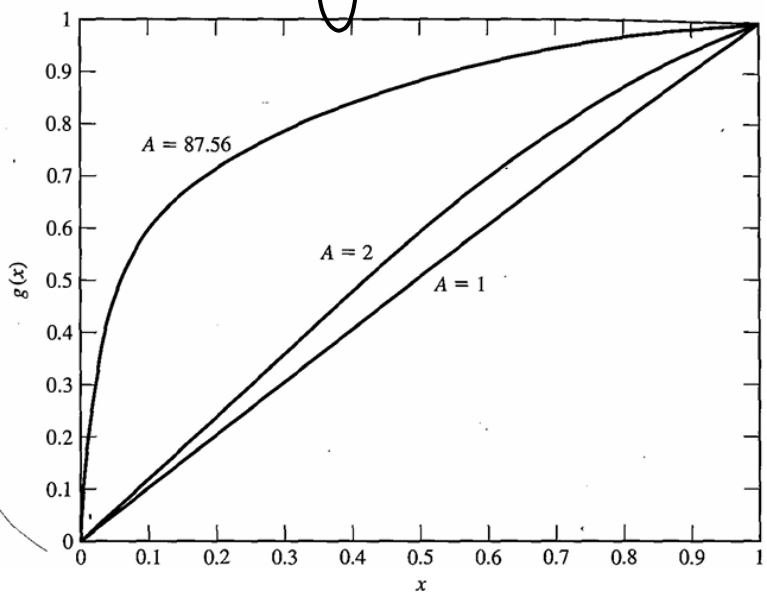


Figure 7.9 A graph of μ-law compander characteristics.

(ii) A - law companding:

$$g(x) = \frac{1 + \log A|x|}{1 + \log A} \operatorname{sgn}(x)$$



7.4.2 : Differential PCM :

•) After sampling information signal, each sample quantised independently. This means previous sample has no effect on quantisation of new samples.

•) Sampled values are correlated such that spectrum of the BW is flat. Means that previous & present sample is related i.e., previous value small high probability that present value is small as well

•) DPCM, difference btw 2 adjacent value is quantized, highly correlated

value result in smaller difference. DPCM require fewer levels to quantise it.

Correlation in Sampled Values

Many real-world signals (voice, audio, etc.) are **bandlimited and smooth**, so their sampled values are *correlated*: a sample at time n is often similar to $n-1$.

- Flat spectrum (white noise) → no correlation
- Normal signals (speech, music) → lots of correlation

This means if the previous value is small, it's highly probable that the present value is also small, and so on.

DPCM (Differential PCM): Exploiting Correlation

- **DPCM** takes advantage of this correlation by quantizing **differences** between successive samples, instead of raw sample values:

$$\text{DPCM: Quantize } d(n) = x(n) - x(n-1)$$

- If the samples are highly correlated, **the difference is much smaller** (on average) than the absolute values. Smaller numbers can be represented using fewer bits (quantization levels).

Efficient Encoding

- Smaller differences mean the quantizer can have **fewer levels**, or for the same number of bits, quantization error is **lower**.
- DPCM is **more efficient** than PCM for correlated signals:
  - Less bit-rate needed for same quality.
  - Or, better quality for the same bit-rate.

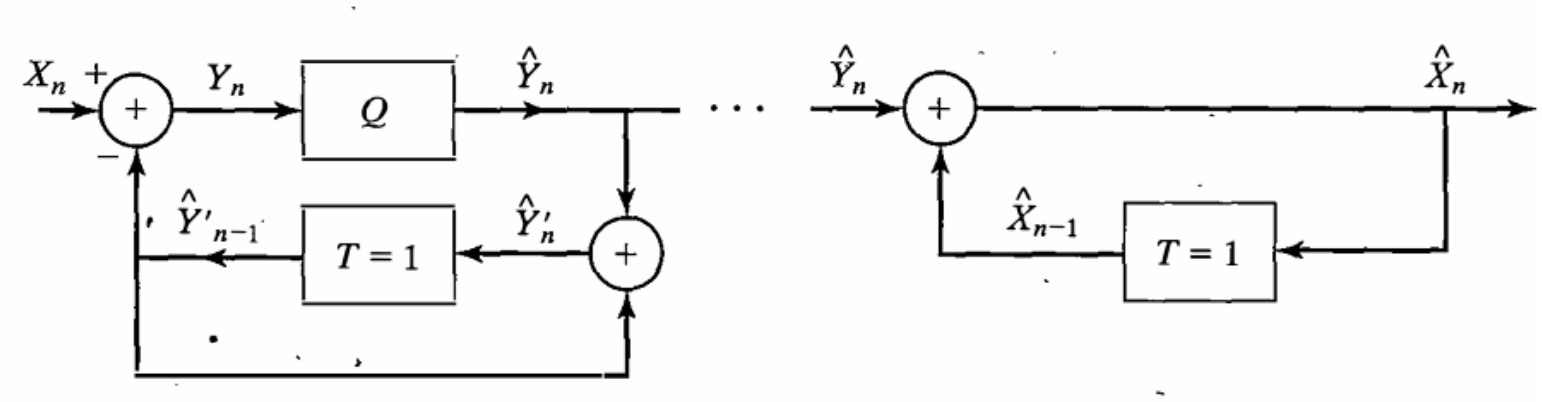


Figure 7.11 A simple DPCM encoder and decoder.

1. Block Diagram Analysis

- Inputs and Outputs:
  - X_n: Current input sample.
  - $\hat{X}'_{n-1}$: Previous estimate of the input (from previous time step).
  - Q: Quantizer that digitizes the difference signal.
  - Y_n: Difference signal sent to quantizer.
  - $\hat{Y}_n$: Quantized difference signal.
  - $T = 1$: Delay element (stores previous value).
  - $\hat{X}_n$: Reconstructed output at receiver.

2. Key Equations and Their Significance

a. Input to Quantizer

$$Y_n = X_n - \hat{X}'_{n-1}$$

- Instead of quantizing only the difference between current and previous sample ($X_n - X_{n-1}$), DPCM quantizes the difference between the current input and **previous prediction** ($X_n - \hat{X}'_{n-1}$).
- This approach is advantageous because $\hat{X}'_{n-1}$ is "closely related" to X_{n-1}; thus, the difference is small (due to signal correlation) and quantization errors do not accumulate uncontrollably.

b. Prediction at the Encoder

$$\hat{X}'_n = \hat{Y}_n + \hat{X}'_{n-1}$$

- The prediction of the next sample is generated by **adding the quantized difference** ($\hat{Y}_n$) **to the previous prediction** ($\hat{X}'_{n-1}$).
- This recursive relation means each new prediction builds on the previous one and the newly quantized difference.

c. Quantization Error Analysis

$$\hat{Y}_n - Y_n = \hat{X}_n - X_n$$

- The **error between quantized and actual difference** ($\hat{Y}_n - Y_n$) **is exactly the error between the reconstructed output and the actual input** ($\hat{X}_n - X_n$).
- This is crucial: it shows that, **if the initial conditions are aligned**, the quantization error will propagate consistently, preventing error accumulation due to feedback loop.

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d. Reconstruction at the Decoder (Receiver End)

$$\hat{X}_n = \hat{Y}_n + \hat{X}_{n-1}$$

- The decoder reconstructs the output by **adding the quantized difference** ($\hat{Y}_n$) **to its own previous output** ($\hat{X}_{n-1}$).
- If both encoder and decoder start with the same initial condition, their predictions will match at each step, and the reconstructed output follows the actual signal closely, with quantization error as the only difference.

Process	Key Equation	Physical Meaning
Quantizer Input	$Y_n = X_n - \hat{X}'_{n-1}$	Input is difference from previous prediction
Encoder Prediction	$\hat{X}'_n = \hat{Y}_n + \hat{X}'_{n-1}$	Prediction updates using quantized difference
Decoder Reconstruction	$\hat{X}_n = \hat{Y}_n + \hat{X}_{n-1}$	Output reconstructed from quantized diff and previous
Error Propagation	$\hat{Y}_n - Y_n = \hat{X}_n - X_n$	Quantization error is direct output error

7.4.3: Delta Modulation: